Lecture 8

Monday, February 3, 2020 6:21 AM

Power series.

Recap. • Given a power series $\sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha$, the domain of convergence D is an open Reinhardt domain such that $\sum_\alpha a_\alpha z^\alpha$ converges normally in D to a holomorphic function $f(z)$.

• The open set $D^* = \{\xi \in \mathbb{R}^n : e^\xi = (e^{\xi_1}, \dots, e^{\xi_n}) \in D\}$ has the properties:

① D^* is convex;

② If $\xi \in D^*$ and $\eta_j \le \xi_j$, $j = 1, \dots, n$, then $\eta \in D^*$

Def 1. A Reinhard domain $\Omega \subseteq \mathbb{C}^n$ containing 0 is said to be logarithmically convex if Ω^* satisfies ① and ② above.

Thm 3. Let $\Omega \subseteq \mathbb{C}^n$ be connected Reinhardt domain with $0 \in \Omega$. If f is holom. in Ω, then $\exists$ unique power series $\sum_\alpha a_\alpha z^\alpha$ converging normally in Ω to f.

Pf. We first note that uniqueness is clear, since differentiation $\Rightarrow a_\alpha = \frac{1}{\alpha!} f_{z^\alpha}(0)$. Consider the following exhaustion:

$\forall \varepsilon > 0$, $\Omega_\varepsilon := \{z \in \Omega : d(z, \mathbb{C}^n \setminus \Omega) > \varepsilon |z|\}$,

Also Reinhardt! (Ex.)

reasonable any norm on $\mathbb{C}^n$, e.g. $|z| = (\sum |z_j|^2)^{1/2}$.

and let Ω_ε' be the component that contains 0. Since Ω connected $\Rightarrow \Omega = \bigcup_{\varepsilon > 0} \Omega_\varepsilon'$ (and $\Omega_{\varepsilon_1}' \subseteq \Omega_{\varepsilon_2}'$ if $\varepsilon_2 \le \varepsilon_1$).

Thus, suffices to prove the statements in Thm 3 in Ω_ε', small $\varepsilon > 0$.

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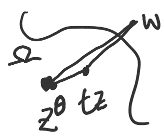
Since $0 \in \Omega'_\varepsilon$, there are $z \in \Omega'_\varepsilon$ s.t. $z_j \neq 0$, $j = 1, \dots, n$, and s.t. the closed polydisk $\mathbb{D}^n = \{\zeta \in \mathbb{C}^n : |\zeta_j| \leq (1+\varepsilon)|z_j|\} \subseteq \Omega'_\varepsilon$. By CF $\Rightarrow$

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{D}^n} \frac{f(\zeta)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n = \left\{ \begin{matrix} \zeta_j = t_j z_j \\ d\zeta_j = z_j \, dt_j \end{matrix} \;\middle|\; \underbrace{|t_j| = 1+\varepsilon}_{T^n} \right\}$$

$$= \frac{1}{(2\pi i)^n} \int_{\partial_0 T^n} \frac{f(tz)}{(t_1 - 1)\cdots(t_n - 1)} \, dt_1 \cdots dt_n.$$

Next, we note that for <u>any</u> $z \in \Omega'_\varepsilon$ and $t \in \partial_0 T^n$, $tz = (1+\varepsilon)\underbrace{(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n)}_{z^\theta}$ and $z^\theta \in \Omega_\varepsilon$ by Reinhardt prop. and $|tz - z^\theta| = \varepsilon |z|$

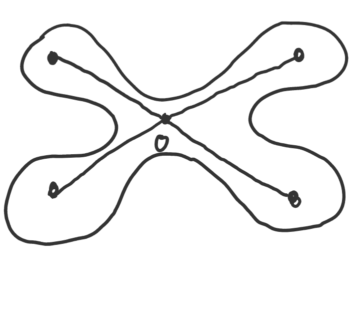
$$d(tz, \mathbb{C}^n \setminus \Omega) \geq d(z^\theta, \mathbb{C}^n \setminus \Omega) - |tz - z^\theta| > \varepsilon|z| - \varepsilon|z| = 0 \Rightarrow tz \in \Omega.$$



Thus, the integral $\dfrac{1}{(2\pi i)^n} \displaystyle\int_{\partial_0 T^n} \frac{f(tz)}{(t_1 - 1)\cdots(t_n - 1)} \, dt_1 \cdots dt_n$ is defined, and hence holom. (by moving $\bar\partial$ inside $\int$) for $z \in \Omega'_\varepsilon$.

Since Ω'_ε connected and integral $\equiv f$ for z near 0, the integral represents $f(z)$ in <u>all</u> of Ω'_ε. (Note that tz need <u>not</u> be in Ω for <u>all</u> $t \in T^n$, only for $t \in \partial_0 T_n$.)

"<u>Ex</u>."



"Ω"

Now, for $t\in\partial_0 T^n$, $\frac{1}{(t_1-1)\cdots(t_n-1)} = \frac{1}{t_1\cdots t_n}\frac{1}{(1-\frac{1}{t_1})\cdots(1-\frac{1}{t_n})} = \sum_\alpha \frac{1}{t_1^{1+\alpha_1}}\cdots\frac{1}{t_n^{1+\alpha_n}}$

w/ normal convergence in $\partial_0 T^n$. We obtain:

$$f(z) = \sum_\alpha \underbrace{\frac{1}{(2\pi i)^n}\int_{\partial_0 T^n} \frac{f(tz)}{t_1^{\alpha_1+1}\cdots t_n^{\alpha_n+1}}\,dt}_{f_\alpha(z) \text{ holom. in } \Omega'_\varepsilon}$$

If $K\subset\subset\Omega'_\varepsilon$ compact, then $\{tz\in\Omega : z\in K, t\in\partial_0 T^n\}\subset\subset\Omega$ is compact $\Rightarrow \sup_{t\in\partial_0 T, z\in K}|f(tz)|\le M$, $|t_1^{\alpha_1+1}\cdots t_n^{\alpha_n+1}| = (1+\varepsilon)^{n+|\alpha|}$

$\Rightarrow \sum_\alpha$ above converges normally in Ω'_ε. Moreover, for $z\in\Omega'_\varepsilon$ and $z_j\neq 0$, we find

$$f_\alpha(z) = \left\{\begin{matrix}\zeta = tz\\ d\zeta = z\,dt\end{matrix}\right\} = \frac{1}{(2\pi i)^n}\int_{|\zeta_j|=(1+\varepsilon)|z_j|} \frac{f(\zeta)}{(\zeta_1/z_1)^{\alpha_1+1}\cdots(\zeta_n/z_n)^{\alpha_n+1}}\, z_1\cdots z_n\, d\zeta_1\cdots d\zeta_n$$

$$= z^\alpha \underbrace{\frac{1}{(2\pi i)^n}\int \frac{f(\zeta)\,d\zeta_1\cdots d\zeta_n}{\zeta_1^{\alpha_1+1}\cdots\zeta_n^{\alpha_n+1}}\, d\zeta_1\cdots d\zeta_n}$$

$= \frac{1}{\alpha!} f_{z^\alpha}(0)$ by (differentiated) CF applied at $z=0$, when $|z|$ is so small that $\{\zeta : |\zeta_j|\le|z_j|\}$ is contained in Ω. Again by connectedness, we conclude $f_\alpha(z) = \frac{1}{\alpha!} f_{z^\alpha}(0)\, z^\alpha$ in Ω'_ε.

is contained in

we conclude $f_\alpha(z) = \frac{1}{\alpha!} f_{z^\alpha}(0)\, z^\alpha$ in Ω'_ε.

This completes the pf. ☒

Thus, for any connected Reinhardt domain $\Omega \subseteq \mathbb{C}^n$ and any f holom. in Ω the domain of convergence D of the Taylor series of f contains Ω! But Ω need not be log convex, whereas D always is.

Note that if Ω_β, $\beta \in B$ (any index set), are log convex Reinhardt domains, then the interior of $\bigcap_\beta \Omega_\beta$ is also log convex Reinhardt domain (possibly empty).

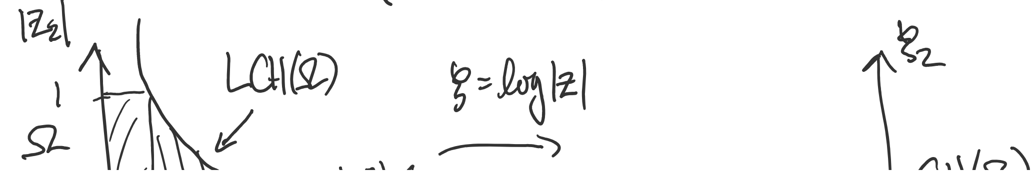
<u>Def 4</u>. If Ω is Reinhardt, let $LCH(\Omega)$ be the interior of the intersection of all log convex Reinhardt domains that contain Ω ($\mathbb{C}^n$ is always among them).

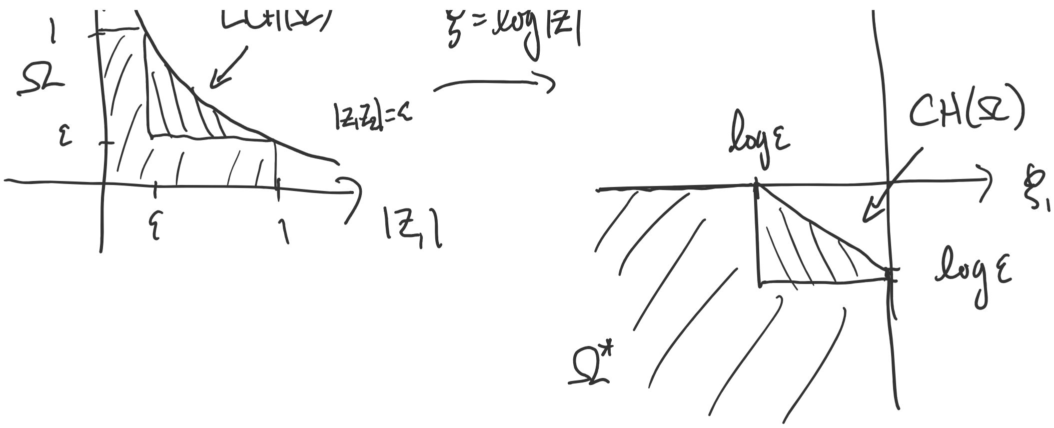
<u>Thm 4</u>. Let Ω be connected Reinhardt domain w/ $0 \in \Omega$. Then, every holom. function f in Ω extends to a holom. function F in $LCH(\Omega)$.

<u>Pf</u>: Direct consequence of Thm 2 and Thm 3. ☒

<u>Ex</u>: $\Omega = \{ z \in \mathbb{C}^2 : \max(|z_1|, |z_2|) < 1, \min(|z_1|, |z_2|) < \varepsilon \} \Rightarrow$

$LCH(\Omega) = \{ z : \max(|z_1|, |z_2|, |z_1 z_2|/\varepsilon) < 1 \}$




CH(Ω)
ξ = log|z|
Ω
|z₁z₂|=ε
ε
1
|z₁|
log ε
CH(Ω)
ξ₁
log ε
Ω*